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The analysis of nonlinear density-wave oscillations in boiling channels

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Thermally induced flow instabilities in uniformly heated boiling channels have been studied analytically. The classical homogeneous equilibrium model was used. This distributed model was transformed into an integrodifferential equation for inlet velocity. A linear analysis showed interesting features (i.e. islands of instability) of the marginal stability boundary which appear when the effects of gravity and friction were systematically considered. A quasilinear Hopf-bifurcation analysis, valid near the marginal-stability boundaries, gives the amplitude and frequency of limit-cycle oscillations that can appear on the unstable side of the boundary. The analysis also shows cases where a finite-amplitude perturbation can cause a divergent instability on the stable side of the linear-stability boundary.

1. Introduction

The hydrodynamic stability of a heated boiling channel is of great practical importance in the design and operation of process equipment in the chemical and power industry. An instability may cause a divergent evolution (oscillating or not), leading to mechanical damage or a violation of thermal limits. Of the various types of flow instability to which two-phase systems are prone, the most common is a relatively low-frequency instability called a density-wave oscillation. Consequently, starting with the work of Serov (1953), this instability type has been studied extensively.

Most previous analytical work (Bouré 1965; Ishii 1971; Yadigaroglu 1978) has been confined to a linear stability analysis of the threshold of instability. Ishii (1971), Yadigaroglu (1978) and Achard, Drew & Lahey (1981) have dealt with the effects of various modelling assumptions on this instability threshold. Krishnan, Atkinson & Friedly (1980) have attempted a nonlinear stability analysis. Such analyses give information about what happens once the linear-stability threshold has been crossed. These authors treated a model that neglected fluid inertia, and was dominated by outlet friction. Recently, Atkinson & Friedly (1982) have shown that this model does not predict limit-cycle oscillations, even though they are observed in experimental situations where the model approximations should be valid (Akinjiola & Friedly 1982). They concluded that the model should be modified to include heated-wall dynamics. In contrast with Friedly's model, we have included fluid inertia. As a consequence our model is able to predict limit-cycle response in a boiling channel without considering local losses and heated-wall dynamics.

A comprehensive linear stability study has been given by Achard *et al.* (1981). By performing a systematic study of the linear-stability boundary, 'islands of instability' were found in the nominally stable operating region. In addition, it was shown that the linearized stability model predicts excursive (i.e. Ledinegg) instability for the zero-frequency limit.

Our study of nonlinear effects is based on the results obtained from linear analysis, and is concerned with whether nonlinear solutions can bifurcate; that is, whether there exist two (or more) solutions corresponding to a given set of external boundary conditions. In particular, we have studied the possibility that a periodic solution may exist near a steady-state solution, and whether such periodic solutions are stable. This bifurcation, or branching, of a periodic solution from a steady-state solution is often called a Hopf bifurcation (Hopf 1942). Linear theory gives the threshold to the onset of instability for infinitesimal disturbances. The nonlinear analysis gives a sufficient condition for stability for finite disturbances, in that it predicts a threshold amplitude above which finite-amplitude oscillations may grow. Moreover, it predicts conditions under which finite-amplitude oscillations may appear.

The model used in this study was chosen to give a compromise between realism and calculational practicality. The heated channel under consideration was assumed to be operating subject to constant-pressure-drop (i.e. parallel-channel) boundary conditions. A homogeneous equilibrium model was used, the heat flux was assumed to be constant and uniform, and the inlet subcooling was held constant.

These conditions are appropriate for describing the operation of a system consisting of many identical heated boiling channels connecting two large plena. In this configuration, the applied pressure drop across any channel is approximately constant, and the mixing that occurs in the lower plenum keeps the inlet temperature approximately constant.

It will be shown in this study that linear stability evaluations of a boiling channel may be nonconservative with respect to finite-amplitude disturbance-induced instabilities in the subcritical case, and conservative with respect to limit-cycle oscillations in the supercritical case.

2. Lumped parameter modelling

The mathematical representation of a heated boiling channel operating at steady state, the stability of which we are studying, is shown in figure 1. Four operating parameters, or boundary conditions, characterize the system:

(i) the heat flux $\bar{q}''_0$, assumed here to be uniform for the sake of simplicity;

(ii) the pressure level p;

(iii) the impressed pressure drop Δp_{ex} across the heated channel.

(iv) the inlet temperature T_{i} or equivalently the inlet subcooling Δh_{sub}.

The boiling channel can be divided into two regions: the single-phase and the two-phase region. For each region, the state variables describing the system are the velocity, pressure and enthalpy.

It was convenient to non-dimensionalize the model using the heated length L_{H}, the time ν for the liquid to lose its inlet subcooling, and the inlet velocity v_0 corresponding to the first appearance of bulk boiling at the exit.

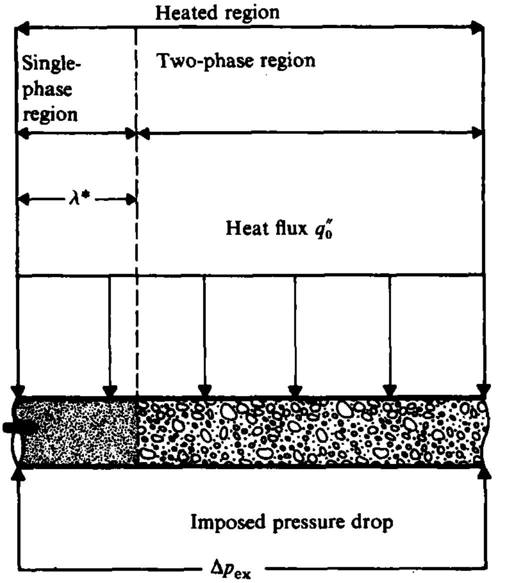

FIGURE 1. Schematic of a typical boiling channel.

The resulting dimensionless forms are

$$Eu = \frac{\Delta p_0}{\rho_v v_0^2} \quad \text{(Euler number)}, \tag{1}$$

$$Fr = \frac{v_0^2}{gL_H} \quad \text{(Froude number)}, \tag{2}$$

$$\Lambda = \frac{fL_H}{2D_H} \quad \text{(friction number)}, \tag{3}$$

$$N_{sub} = \frac{v_{fg}(h_f - h_i)}{v_f h_{fg}} \quad \text{(subcooling number)}, \tag{4}$$

$$\tilde{j} = j_0/v_0 \quad \text{(velocity number)}, \tag{5}$$

where v_0 is the inlet velocity when the boiling boundary is at the outlet of the channel,

$$v_0 = \frac{\bar{q}_0'' P_H v_f L_H}{A_{xs}(h_f - h_i)},$$

and Δp_0 is a pressure-drop scale, which will be defined later, $\bar{q}_0''$ is the heat flux, ρ_k is the density of phase k, L_H and D_H are the heated length and hydraulic diameter of the channel respectively, f is the friction factor, $v_f = 1/\rho_f$, $v_{fg} = 1/\rho_g - 1/\rho_f$, h_i is the inlet enthalpy, h_f is the saturation enthalpy of the liquid, h_g is the saturation enthalpy of the vapour and $h_{fg} = h_g - h_f$. Also, A_{xs} is the cross-sectional area of the channel and P_H is the heated perimeter.

We now let $\lambda(t) \equiv \lambda^*(t)/L_H$ be the dimensionless distance from the inlet to the boiling boundary (Lahey & Moody 1977). If we spatially integrate the energy equation along the single-phase region $0 \leqslant z \leqslant \lambda(t)$, the dimensionless system becomes

$$\langle j \rangle = j, \tag{6}$$

$$-Eu\frac{\partial p}{\partial z} = \frac{\mathrm{d}j}{\mathrm{d}t} + \Lambda j^2 + Fr^{-1}, \tag{7}$$

$$\lambda(t) = \int_{t-1}^{t} j(t')\,\mathrm{d}t', \tag{8}$$

where the angular brackets $\langle\ \rangle$ represent an average over the cross-section of the channel.

Similarly, for the two-phase region, $\lambda(t) \leqslant z \leqslant 1$, we have

$$\frac{\partial\langle j \rangle}{\partial z} = N_{\text{sub}}, \tag{9}$$

$$-Eu\frac{\partial p}{\partial z} = \langle\rho_{\mathrm{H}}\rangle\left(\frac{\partial\langle j\rangle}{\partial t} + \langle j\rangle\frac{\partial\langle j\rangle}{\partial z}\right) + \Lambda\langle\rho_{\mathrm{H}}\rangle\langle j\rangle^2 + Fr^{-1}, \tag{10}$$

$$\frac{\partial\langle X\rangle}{\partial t} + \langle j\rangle\frac{\partial\langle X\rangle}{\partial z} - N_{\text{sub}}\langle X\rangle = N_{\text{sub}}\frac{v_{\mathrm{f}}}{v_{\mathrm{fg}}}, \tag{11}$$

where ρ_{H} is the (homogeneous) density, $\langle j\rangle$ is the two-phase volumetric flux, j is the inlet velocity, p is system pressure and $\langle X\rangle$ denotes the flow quality.

A lumped-parameter system of equations can be derived from this distributed system by integrating (9) and (11) along their characteristics, using the results in the mixture momentum equations (10), and integrating (7) and (10) along the heated length. We thus obtain the pressure drop in the single-phase portion of the channel (Achard *et al.* 1981) as

$$\Delta p_{1\phi} = \int_{z=0}^{z=\lambda(t)}\left(-\frac{\partial p}{\partial z}\right)\mathrm{d}z = \frac{\lambda(t)}{Eu}\left[\frac{\mathrm{d}j}{\mathrm{d}t} + \Lambda j^2 + Fr^{-1}\right]. \tag{12}$$

Similarly, for the two-phase portion we have

$$\Delta p_{2\phi} = \int_{z=\lambda(t)}^{z=1}\left(-\frac{\partial p}{\partial z}\right)\mathrm{d}z = \frac{1}{Eu}\int_{\lambda(t)}^{1}\langle\rho_{\mathrm{H}}\rangle\left(\frac{\partial\langle j\rangle}{\partial t} + \langle j\rangle\frac{\partial\langle j\rangle}{\partial z} + \Lambda\langle j\rangle^2 + Fr^{-1}\right)\mathrm{d}z. \tag{13}$$

A further simplification is achieved if we change the variable of integration from space to time difference ($t' = t - t_0$), and combine (9) with (13) to obtain

$$\begin{aligned}\Delta p_{2\phi} = {} & \frac{1}{Eu}\left(\frac{\mathrm{d}j}{\mathrm{d}t} + N_{\text{sub}}\,j(t-1) + \Lambda j^2 + Fr^{-1}\right)\int_0^{\tau(t)} j(t-t'-1)\,\mathrm{d}t' \\ & + \frac{1}{Eu}(2\Lambda N_{\text{sub}}\,j + N_{\text{sub}}^2)\int_0^{\tau(t)}(z-\lambda(t))\,j(t-t'-1)\,\mathrm{d}t' \\ & + \frac{1}{Eu}N_{\text{sub}}^2\,\Lambda\int_0^{\tau(t)}(z-\lambda(t))^2\,j(t-t'-1)\,\mathrm{d}t'.\end{aligned} \tag{14}$$

The total pressure drop can now be obtained by adding (12) and (14), yielding

$$\begin{aligned}(\lambda + I_0)\frac{\mathrm{d}j}{\mathrm{d}t} = {} & N_{\text{sub}}\,I_0\,G_{\mathrm{b}} - N_{\text{sub}}^2\,J_1 - Fr^{-1}(\lambda + I_0) \\ & - \Lambda(j^2(\lambda + I_0) + 2N_{\text{sub}}\,jJ_1 + N_{\text{sub}}^2\,K_2) + Eu\,\Delta p_{\text{ex}},\end{aligned} \tag{15}$$

where Δp_{ex} is the imposed dimensionless pressure drop ($0 \leqslant \Delta p_{\mathrm{ex}} \leqslant 1$) and

$$G_{\mathrm{b}}(t) \equiv j(t-1), \tag{16a}$$

$$\lambda(t) = \int_{t-1}^{t} j(t')\,\mathrm{d}t'' = \int_0^1 j(t-t'')\,\mathrm{d}t''. \tag{16b}$$

$$I_0 \equiv \int_0^{\tau(t)} G_{\mathrm{b}}(t-t')\,\mathrm{d}t', \tag{16c}$$

$$J_1 \equiv \int_0^{\tau(t)} \int_0^{t'} \mathrm{e}^{N_{\mathrm{sub}}t''} G_{\mathrm{b}}(t-t'')\,\mathrm{d}t''\, G_{\mathrm{b}}(t-t')\,\mathrm{d}t', \tag{16d}$$

$$K_2 \equiv \int_0^{\tau(t)} \left\{\int_0^{t'} \mathrm{e}^{N_{\mathrm{sub}}t''} G_{\mathrm{b}}(t-t'')\,\mathrm{d}t''\right\}^2 G_{\mathrm{b}}(t-t')\,\mathrm{d}t'. \tag{16e}$$

The lumped system, given by (15)–(16*e*), is an integrodifferential equation which governs the behaviour of the inlet velocity $j(t)$. The solution $j(t)$ of this system depends on the history of $j(t-t')$ for delays t' satisfying $0 < t' < 1+\tau(t)$.

The relationship between the steady-state pressure drop Δp_{ex} and the steady-state inlet-velocity ratio $\tilde{j}$ is given by (15) as

$$\Delta p_{\mathrm{ex}} = \{N_{\mathrm{sub}}\tilde{j}(1-\tilde{j}) + Fr^{-1}\tilde{j}(1+\tilde{\tau}) + \Lambda[\tilde{j}^2 + \tfrac{1}{2}\tilde{j}N_{\mathrm{sub}}(1-\tilde{j})^2]\}/Eu. \tag{17}$$

This relation gives two results. First, it gives the pressure-drop scale Δp_0 used in the Euler number (1). This is obtained by putting $\Delta p_{\mathrm{ex}} = 1$ and $\tilde{j} = 1$. The Euler number is then a function of Λ, N_{sub} and Fr, and is thus not an independent dimensionless number. Secondly, it gives the dimensionless pressure drop Δp_{ex} given values of N_{sub}, Fr, Λ and $\tilde{j}$. So a steady state can be specified either by the vector of parameters $(N_{\mathrm{sub}}, Fr, \Lambda, \Delta p_{\mathrm{ex}})^{\mathrm{T}}$, or equivalently by $(N_{\mathrm{sub}}, Fr, \Lambda, \tilde{j})^{\mathrm{T}}$. The second vector is preferred, because when Δp_{ex} is given by (17) we cannot always determine $\tilde{j}$ uniquely. We denote the vector of interest by

$$\boldsymbol{\zeta} = (N_{\mathrm{sub}}, Fr, \Lambda, \tilde{j})^{\mathrm{T}}. \tag{18}$$

For the linear and quasilinear stability analyses we must expand (15) and (16) about the steady state. To this end we write $j = \tilde{j} + \delta j$. In Hopf-bifurcation analysis (Hopf 1942; Howard & Koeppel 1976), it is necessary to retain terms through third-order (δj^3) perturbations. Let us summarize the perturbation procedure used in this study. First a linear stability analysis was performed to determine the values of the various parameters at the critical point (i.e. the point where the solution changes from stable to unstable). To use Hopf bifurcation, the eigenvalue of the linear problem must have a non-zero imaginary part so that the solution is oscillatory instead of excursive. Thus, at the critical point, the linearized equations have a periodic solution.

A perturbation expansion is assumed for the periodic solution, where the small parameter is a suitably defined amplitude. In order to determine where in parameter space this periodic solution occurs, one parameter, chosen as the critical parameter, has its amplitude expanded in a power series. The frequency is also expanded in the same way. The relation between the critical parameter and the amplitude is determined by the requirement that resonances not occur at any order. To third order, this gives the relation between the first term in the expansion of the critical parameter and the disturbance amplitude. Expansions up to and including the cubic part of (15) and (16) have been performed (Achard, Drew & Lahey 1980).

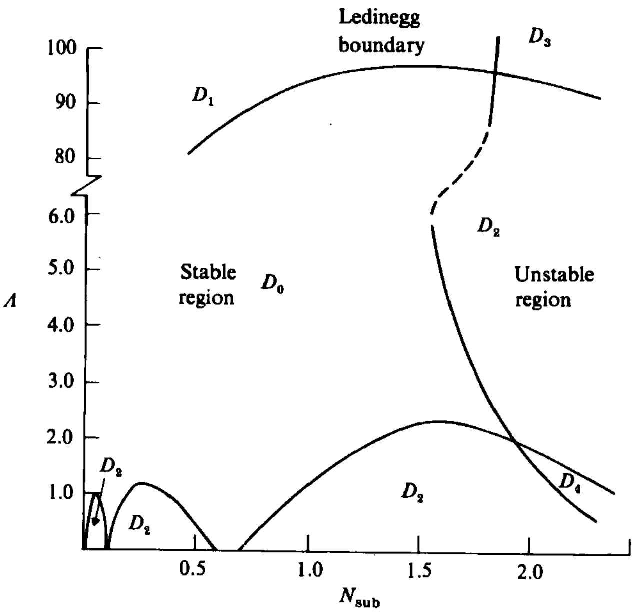

FIGURE 2. Linear-stability boundaries ($Fr^{-1} = 50$, $= 0.3$).

3. Linear stability analysis

The linear part of the perturbation expansion was Laplace-transformed to yield the following characteristic equation (Achard *et al.* 1981):

$$\Phi(s) \equiv a_0 + a_1 s + a_2 s^2 + \frac{e^{-s}}{(2N_{sub}-s)(N_{sub}-s)^2}[b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4] + \frac{e^{-(1+\tilde{\tau})s}}{(2N_{sub}-s)(N_{sub}-s)^2}[c_0 + c_1 s + c_2 s^2 + c_3 s^3] = 0. \quad (19)$$

The coefficients are given by (Achard *et al.* 1981)

$$a_i = \hat{a}_i + \Lambda\hat{\hat{a}}_i, \quad b_i = \hat{b}_i + \Lambda\hat{\hat{b}}_i, \quad c_i = \hat{c}_i + \Lambda\hat{\hat{c}}_i \quad (20)$$

and

$$\left.\begin{aligned}
&\hat{a}_0 = N_{sub}\tilde{\jmath} - Fr^{-1}(1 - e^{-N_{sub}\tilde{\tau}}), \quad \hat{\hat{a}}_0 = \tilde{\jmath}(e^{N_{sub}\tilde{\tau}} - 1),\\
&\hat{\hat{a}}_1 = -2\tilde{\jmath}, \quad \hat{a}_2 = -\tilde{\jmath}(1+\tilde{\tau}),\\
&\hat{b}_0 = -2N^3_{sub}(N_{sub}\tilde{\jmath} + Fr^{-1}e^{-N_{sub}\tilde{\tau}}), \quad \hat{\hat{b}}_0 = 2N^3_{sub}\tilde{\jmath}^2(1 - e^{N_{sub}\tilde{\tau}}),\\
&\hat{b}_1 = 7N^3_{sub}\tilde{\jmath} + 2Fr^{-1}N^2_{sub}e^{-N_{sub}\tilde{\tau}}, \quad \hat{\hat{b}}_1 = 7N^3_{sub}\tilde{\jmath}(1-\tilde{\jmath}) + N^2_{sub}\tilde{\jmath}^2,\\
&\hat{b}_2 = 2N^3_{sub}\tilde{\jmath}\tilde{\tau} - 5N^2_{sub}\tilde{\jmath} - N_{sub}e^{-N_{sub}\tilde{\tau}}Fr^{-1}, \quad \hat{\hat{b}}_2 = N_{sub}\tilde{\jmath}^2(6 - 7e^{N_{sub}\tilde{\tau}}),\\
&\hat{b}_3 = N_{sub}\tilde{\jmath}(1 - 3N_{sub}\tilde{\tau}), \quad \hat{\hat{b}}_3 = 2N_{sub}\tilde{\jmath}(1-\tilde{\jmath}), \quad \hat{b}_4 = N_{sub}\tilde{\jmath}\tilde{\tau},\\
&\hat{c}_0 = 2N^3_{sub}Fr^{-1}, \quad \hat{c}_1 = -2N^2_{sub}Fr^{-1} - 2N^3_{sub}\tilde{\jmath}e^{N_{sub}\tilde{\tau}}, \quad \hat{\hat{c}}_1 = -N^2_{sub}\tilde{\jmath}^2e^{2N_{sub}\tilde{\tau}},\\
&\hat{c}_2 = N^2_{sub}\tilde{\jmath}e^{N_{sub}\tilde{\tau}} + N_{sub}Fr^{-1}, \quad \hat{\hat{c}}_2 = N_{sub}\tilde{\jmath}^2e^{2N_{sub}\tilde{\tau}},\\
&\hat{a}_1 = \hat{\hat{a}}_2 = \hat{\hat{b}}_4 = \hat{\hat{c}}_0 = 0.
\end{aligned}\right\} \quad (21)$$

The roots of (19) determine the stability of the system. Figure 2 is a typical linear-stability map. Regions having k roots with Re $(s) > 0$ are denoted by D_k. Hence the region of linear stability is denoted by D_0.

Figure 2 shows several 'islands of instability' in the nominally stable region of the map. These 'islands' appear when gravity effects are important ($Fr \ll 1$), when friction effects are not too high ($\Lambda \leqslant 1$), and when the length of the two-phase region is significant. These 'islands' were suggested by Yadigaroglu & Bergles (1972) in an experimental study of density-wave oscillations.

It is interesting to note in figure 2 that the Ledinegg, or excursive, stability boundary appears at relatively high values of Λ.

Figure 2 shows four possibilities for a heated channel to go from stable to unstable operation. First, if we allow the parameters to vary so that we go from region D_0 to D_1, this indicates the onset of an excursive instability. In so doing, only one root crosses the imaginary axis, and it is real. If we allow the parameters to vary so that we go from D_0 to D_2, then two complex-conjugate roots cross the imaginary axis. This implies that the instability diverges harmonically. There are two pathological cases also shown, namely, going from D_0 to D_3 or D_4 respectively. These cases are unlikely to be encountered in practice, since, even if the intersection points were known exactly, it would be difficult to vary the parameters sufficiently accurately to cross from D_0 into either D_3 or D_4 without going through one of the other regions (D_1 or D_2) first.

4. Nonlinear analysis

Hopf-bifurcation theory (Hopf 1942) describes the evolution of perturbations that are small, but finite, for flow conditions that are near the marginal-stability boundary. This theory shows the existence of families of periodic solutions for the perturbations in a narrow strip either on the stable side, or the unstable side, of the marginal-stability boundary. The case when the periodic solution lies on the stable side of the stability boundary is called a subcritical bifurcation, and the case when the periodic solution lies on the unstable side is called a supercritical bifurcation. When operating in the stable region, the theory shows that, for a subcritical case, sufficiently large perturbations will diverge from the steady state; while, in the supercritical case, all small but finite-amplitude perturbations decay to zero in the stable region. For operation in the unstable region in the subcritical case, all perturbations diverge from the equilibrium. In contrast, in the supercritical case, the periodic solution is stable in the region of linear instability, and thus all perturbations eventually evolve to a limit cycle.

In spite of the inherent limitations of linear analysis, it is often compared with experimental stability thresholds measured in the presence of a fairly high level of noise. Any discrepancies between theoretical predictions and experimental results are usually attributed to simplifications in the analytical model used. While that possibility is always likely, other possibilities exist; indeed, in the case of a subcritical bifurcation, the linear-stability boundary may be quite far from the actual stability boundary for noisy flows.

One of the purposes of this study was to obtain a more realistic stability threshold. It is significant to note that a quasilinear analysis is more stringent than the linear theory in the subcritical case. Another purpose of this study was to predict the amplitude of any resultant density-wave oscillations. The predicted amplitude can be compared with existing data. Finite-amplitude density waves can occur on the unstable side of the linear-stability boundary. These bounded oscillations (i.e. limit cycles) may be acceptable as normal operating points. In this regard, it is significant to note that, in the supercritical case, quasilinear analysis may be less stringent than linear analysis.

The model that we have derived in (15)–(16e) is a complicated integrodifferential equation. For this discussion, we shall represent it in the general form,

$$\mathrm{F}(\zeta, \delta X(t), \delta \dot{X}(t), \delta X(t-t')) = 0, \tag{22}$$

where ζ represents the vector of parameters given in (18), specifying a steady-state flow.

The solution vector $\delta X(t)$ represents the perturbation of the two-phase flow parameters:

$$\delta X = (\delta j_i, \delta\lambda, \delta\tau, \delta I_0, \delta J_1, \delta K_2)^{\mathrm{T}}. \tag{23}$$

The trivial solution $\delta X = 0$ is obviously an equilibrium solution of (22). The analysis of the linear stability of this equilibrium solution was discussed in §2. In this section we will consider nonlinear effects. What we wish to do is to determine for each point ζ in parameter space the corresponding amplitude ϵ of the periodic solution. The points ζ should be 'close' to the linear-stability boundary, denoted by $\zeta = \zeta_0$. Thus $\epsilon = \epsilon(\zeta)$, where, by definition, $\epsilon(\zeta_0) = 0$.

By specifying a value for ϵ, say ϵ_1, we can find a curve in the $(N_{\mathrm{sub}}, \Lambda)$-plane where the periodic solution has amplitude ϵ_1. This curve then gives the parameter values corresponding to a limit-cycle solution of amplitude ϵ_1, in the supercritical case, or to the threshold amplitude above which periodic solutions diverge in the subcritical case.

The starting point for a quasilinear analysis is the linear stability problem, which was summarized previously. Let us consider first a supercritical bifurcation. To get a limit cycle, it is necessary that a conjugate pair or roots, $s = \alpha$ and $s = \bar{\alpha}$, crosses the imaginary axis.

Note that $\alpha = \alpha(N_{\mathrm{sub}}, \Lambda)$ and, for the case of interest here, $\alpha(N_{\mathrm{sub}_0}, \Lambda_0)$ is purely imaginary; hence

$$\bar{\alpha}(N_{\mathrm{sub}_0}, \Lambda_0) = -\alpha(N_{\mathrm{sub}_0}, \Lambda_0) \neq 0. \tag{24}$$

Thus the angular frequency ω_0 is defined by

$$\omega_0 = \mathrm{Im}\,[\alpha(N_{\mathrm{sub}_0}, \Lambda_0)] > 0. \tag{25}$$

It is convenient here to use $\zeta_i = \Lambda$ as the critical parameter, since Λ appears linearly in (15)–(16e). Thus we choose

$$\mu = \Lambda - \Lambda_0. \tag{26}$$

When we expand the equation in the small parameter μ we can save considerable work if this choice is made. Also, Λ appears to be appropriate, since at most points on the linear-stability surface an increment in Λ causes the point to move off the surface. If this did not happen, the expansions to be used would be inappropriate, which would be indicated by the inability to calculate the coefficients in the expansions of the critical parameter and frequency. This problem can be understood in terms of the linear-stability boundary plots. Specifically, if there is a point on the curve $\mathrm{Re}\,(s) = 0$ with $\mathrm{Re}\,(\partial\alpha/\partial\mu) = 0$, the expansion procedure fails at that point. At such a point, another choice of the critical parameter is needed.

In Appendix A two mathematical results that will be useful in later analyses are given. The first one is original and shows what can be done if one encounters an inappropriate choice of the critical parameter. The second one is the well-known theorem due to Hopf (1942), which forms the basis of all our calculations. It is basically an existence theorem and justifies the search for a periodic solution for $\delta X(t)$.

			Stability of the equilibrium solution	Existence of a periodic solution	Stability of the periodic stability
			Supercritical		
$\mathrm{Re}\left[\frac{\partial\alpha(0)}{\partial\mu}\right] > 0$	$\mu_{(2)} > 0$	$\mu > 0$	Unstable	Yes	Stable
		$\mu < 0$	Stable	No	—
			Subcritical		
	$\mu_{(2)} < 0$	$\mu > 0$	Unstable	No	—
		$\mu < 0$	Stable	Yes	Unstable
			Subcritical		
$\mathrm{Re}\left[\frac{\partial\alpha(0)}{\partial\mu}\right] < 0$	$\mu_{(2)} > 0$	$\mu > 0$	Stable	Yes	Unstable
		$\mu < 0$	Unstable	No	—
			Supercritical		
	$\mu_{(2)} < 0$	$\mu > 0$	Stable	No	—
		$\mu < 0$	Unstable	Yes	Stable

TABLE 1. Implications of the sign of $\mu_{(2)}$

Some of the consequences of this theorem will be shown for our particular system. For example, it follows directly from the theorem that the expansions of μ and T are of the form

$$\mu = \mu_{(1)}\epsilon + \mu_{(2)}\epsilon^2 + \dots, \tag{27}$$

$$T = 2\pi(1/\omega_0 + \xi_{(1)}\epsilon + \xi_{(2)}\epsilon^2 + \dots), \tag{28}$$

where, necessarily, $\mu_{(1)} = \xi_{(1)} = 0$. The theoretical reason is that the solution must be the same for $\epsilon > 0$ and $\epsilon < 0$. Thus only even powers may appear in (27) and (28). We shall show this shortly, along with the result that $\mu_{(2)} \neq 0$ and $\xi_{(2)} \neq 0$.

A very important by-product of the theorem will also be used. The result concerns the stability of the periodic solution $\mathbf{p}(t, \epsilon)$. The periodic solution will be stable (i.e. it will be a limit cycle) if the corresponding steady solution is unstable. Also, if the steady-state solution is linearly stable, the periodic solution will be unstable. The results are summarized in table 1.

To proceed with the calculation, let us assume that the operator represented by (15) and (16) can be written in the form

$$\mathrm{F}(A, \delta j) = 0. \tag{29}$$

Writing

$$\delta j = \epsilon j \tag{30}$$

and expanding for small δj gives

$$\left.\begin{aligned} 0 &= \mathrm{L}(A, j) + \epsilon\mathrm{Q}(j, j) + \epsilon^2\mathrm{C}(j, j, j) + O(\epsilon^3), \\ 0 &= \mathrm{L}(A_0, j) + (A - A_0)\frac{\partial \mathrm{L}}{\partial A}(A_0, j) + \epsilon\mathrm{Q}(j, j) + \epsilon^2\mathrm{C}(j, j, j) + O(\epsilon^3) + O(A - A_0)^2, \end{aligned}\right\} \tag{31}$$

where L is the linearized version of (15) and (16), and Q and C are the quadratic and cubic terms respectively. They are written as bilinear and trilinear forms; i.e.

$Q(j+h,j) = Q(j,j)+Q(h,j)$. Note that L, $\partial L/\partial \Lambda$, Q and C are operators involving differentiation, delay and integration of j.

Let us now describe the procedure to obtain the periodic solution. In order to do this, it is convenient to 'normalize' the solution so that the period is exactly 2π. Thus we introduce a variable θ, defined by

$$t = \theta(1/\omega_0+\xi). \tag{32}$$

If we now expand j, Λ and ξ in powers of ϵ, we have

$$j = j_{(0)}+\epsilon j_{(1)}+\epsilon^2 j_{(2)}+\ldots, \tag{33}$$

$$\Lambda = \Lambda_0+\epsilon\mu_{(1)}+\epsilon^2\mu_{(2)}+\ldots, \tag{34}$$

$$\xi = \epsilon\xi_{(1)}+\epsilon^2\xi_{(2)}+\ldots. \tag{35}$$

To lowest order in ϵ, (31) gives

$$L(\Lambda_0,j_{(0)}) = 0. \tag{36}$$

Equation (36) is precisely the equation governing the linear stability of the steady state, with a change of time variable from t to θ. Thus we see that $e^{\pm i\theta}$ are solutions. The real-valued solution we wish to consider is

$$j_{(0)} = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta}). \tag{37}$$

This particular solution satisfies the 'initial' conditions

$$\frac{1}{\pi}\int_{-2\pi}^{0} j_{(0)}(\theta)\cos\theta\,d\theta = 1 \tag{38}$$

and

$$\frac{1}{\pi}\int_{-2\pi}^{0} j_{(0)}(\theta)\sin\theta\,d\theta = 0. \tag{39}$$

If we also require that for the higher-order solutions

$$\frac{1}{\pi}\int_{-2\pi}^{0} j_{(k)}(\theta)\cos\theta\,d\theta = \frac{1}{\pi}\int_{-2\pi}^{0} j_{(k)}(\theta)\sin\theta\,d\theta = 0 \quad (k = 1,2,\ldots) \tag{40}$$

then the solution satisfies

$$\frac{1}{\pi}\int_{-2\pi}^{0} \delta j(\theta)\cos\theta\,d\theta = \epsilon, \tag{41}$$

$$\frac{1}{\pi}\int_{-2\pi}^{0} \delta j(\theta)\sin\theta\,d\theta = 0. \tag{42}$$

Note that (41) and (42) define the relation between the imposed history, $\delta j(\theta)$, for $\theta < 0$, and the parameter ϵ, which is a measure of the amplitude of the disturbance.

The equation for $j_{(1)}(\theta)$ is obtained by collecting all the $O(\epsilon^2)$ terms, and then eliminating all the dependent variables except $j_{(1)}$. This leads to

$$L_0(j_{(1)}) = -\mu_{(1)}\frac{\partial L_0}{\partial \mu}(j_{(0)})+\xi_{(1)}\,L_T(j_{(0)})+Q(j_{(0)}), \tag{43}$$

where

$$\frac{\partial L_0}{\partial \mu}(j_0) = \tfrac{1}{2}\{2\tilde{j}\,[e^{i\theta}+e^{-i\theta}]+\tilde{j}^2\,b^{\lambda}_{(0)}\,e^{i\theta}+\overline{b^{\lambda}_{(0)}}\,e^{-i\theta}\}+\tilde{j}^2\{b^{I_0}_{(0)}\,e^{i\theta}+\overline{b^{I_0}_{(0)}}\,e^{-i\theta}\}$$
$$+2N_{sub}\tilde{j}\{b^{J_1}_{(0)}\,e^{i\theta}+\overline{b^{J_1}_{(0)}}\,e^{-i\theta}\}+N^2_{sub}\{b^{K_2}_{(0)}\,e^{i\theta}+\overline{b^{K_2}_{(0)}}\,e^{-i\theta}\}\} \tag{44}$$

and
$$L_T(j_{(0)}) = \frac{1}{2}\left\{\sum_{k=1}^{20} \beta_k\{b^k_{(0)}\, e^{i\theta} + \overline{b^k_{(0)}}\, e^{-i\theta}\}\right\}. \tag{45}$$

The constants β_k and $b^k_{(0)}$ and the forcing function Q are given in Appendix B.

The Fredholm alternative for (43) yields

$$-\mu_{(1)} \operatorname{Re}\left\{\frac{\partial L_0}{\partial \mu}\right\} + \xi_{(1)} \operatorname{Re}\{L_T\} = 0, \tag{46}$$

$$-\mu_{(1)} \operatorname{Im}\left\{\frac{\partial L_0}{\partial \mu}\right\} + \xi_{(1)} \operatorname{Re}\{L_T\} = 0, \tag{47}$$

which is a homogeneous linear system for the unknowns $\mu_{(1)}$ and $\xi_{(1)}$. The solution is trivial, $\mu_{(1)} = \xi_{(1)} = 0$, unless the determinant is zero.

The constant and $e^{\pm 2i\theta}$ terms are needed in the calculation of $\xi_{(2)}$ and $\mu_{(2)}$. They are given in Appendix C.

The values of $\xi_{(2)}$ and $\mu_{(2)}$, the first two non-zero terms in the expansions of $T(\epsilon)$ and $\mu(\epsilon)$, are obtained by imposing the Fredholm alternative condition on the equations of order ϵ^3. This equation is obtained by collecting all the $O(\epsilon^3)$ terms, and then eliminating all the dependent variables, except $j_{(2)}$. This equation can be written as

$$L_0(j_{(2)}) = -\mu_{(2)} \frac{\partial L_0}{\partial \mu}(j_{(0)}) + \xi_{(2)} L_T(j_{(0)}) + S_Q(j_{(0)}, j_{(1)}) + S_C(j_{(0)}). \tag{48}$$

Our purpose at this order is to calculate $\xi_{(2)}$ and $\mu_{(2)}$ using the Fredholm alternative. The first two terms on the right involve $e^{i\theta}$ alone. The others involve a constant, $e^{i2\theta}$, $e^{-i2\theta}$, $e^{i3\theta}$, and $e^{-i3\theta}$ as well as $e^{i\theta}$ and $e^{-i\theta}$. For simplicity, we write

$$S = S_Q(j_{(0)}, j_{(1)}) + S_C(j_{(0)}). \tag{49}$$

The condition for the existence of a periodic solution of (48) then becomes

$$-\mu_{(2)} \operatorname{Re}\left\{\frac{\partial L_0}{\partial \mu}\right\} + \xi_{(2)} \operatorname{Re}\{L_T\} = -\operatorname{Re}\{S\}, \tag{50}$$

$$-\mu_{(2)} \operatorname{Im}\left\{\frac{\partial L_0}{\partial \mu}\right\} + \xi_{(2)} \operatorname{Im}\{L_T\} = -\operatorname{Im}\{S\}. \tag{51}$$

Solving gives

$$\mu_{(2)} = \frac{\operatorname{Im}\{L_T\}\operatorname{Re}\{S\} - \operatorname{Re}\{L_T\}\operatorname{Im}\{S\}}{\operatorname{Re}\left\{\frac{\partial L_0}{\partial \mu}\right\}\operatorname{Im}\{L_T\} - \operatorname{Im}\left\{\frac{\partial L_0}{\partial \mu}\right\}\operatorname{Re}\{L_T\}}, \tag{52}$$

$$\xi_{(2)} = \frac{\operatorname{Im}\left\{\frac{\partial L_0}{\partial \mu}\right\}\operatorname{Re}\{S\} - \operatorname{Re}\left\{\frac{\partial L_0}{\partial \mu}\right\}\operatorname{Im}\{S\}}{\operatorname{Re}\left\{\frac{\partial L_0}{\partial \mu}\right\}\operatorname{Im}\{L_T\} - \operatorname{Im}\left\{\frac{\partial L_0}{\partial \mu}\right\}\operatorname{Re}\{L_T\}}. \tag{53}$$

5. Implications of the nonlinear analysis

The parameter $\mu_{(2)}$ was calculated numerically for each point on the marginal-stability boundary in the (Λ, N_{sub})-plane (Achard *et al.* 1980). Figures 3 and 4 show the results for $Fr^{-1} = 50, \tilde{j} = 0.325$ and $Fr^{-1} = 50.0, \tilde{j} = 0.50$ respectively. Note that there are four amplitudes ϵ shown in each figure. The solid curve is the marginal-

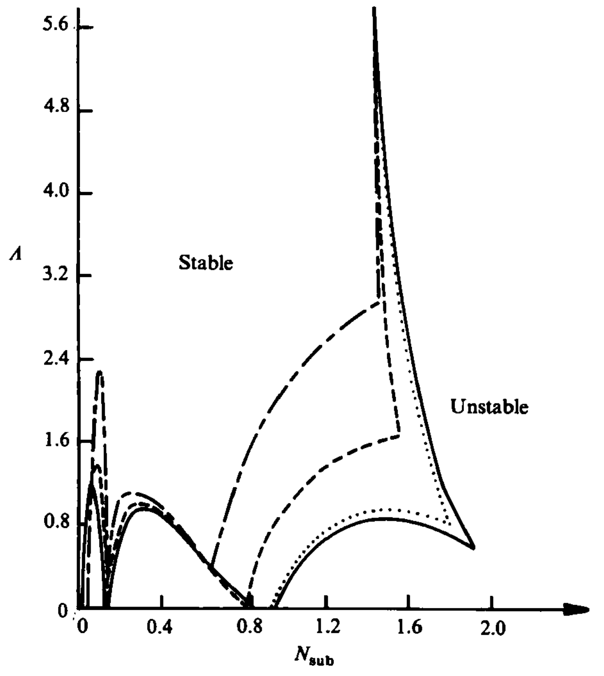

FIGURE 3. Finite-amplitude instability surfaces $j = 0.325$, $Fr^{-1} = 50.0$: ——, $\epsilon = 0$, · · · ·, 0.1; ----, 0.3; — — —, 0.5.

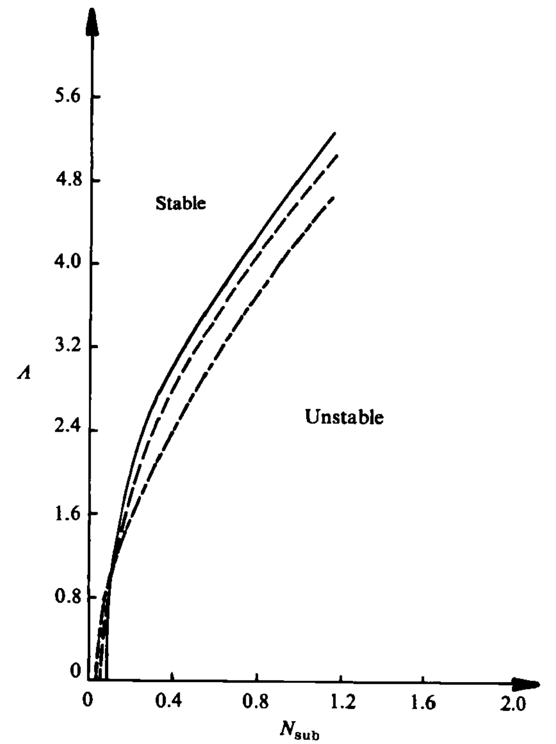

FIGURE 4. Finite-amplitude instability surfaces, $j = 0.50$, $Fr^{-1} = 50.0$: ——, $\epsilon = 0$; ---, 0.3; — — —, 0.5.

stability boundary, while the broken curves indicate the positions in parameter space where various amplitudes of the periodic solution are obtained. When the broken curves lie on the stable side of the marginal-stability curve (the subcritical case), they indicate the location where a disturbance with an amplitude less than the value of ϵ on that curve will decay to zero. In contrast, if a disturbance is present with an amplitude greater than ϵ, that disturbance will grow. When the broken curves lie on the unstable side of the marginal-stability curve (the supercritical case), they indicate the amplitude of the stable limit cycle, to which all disturbances evolve. Regions of both subcritical and supercritical bifurcation can be noted in both figures; however, the most important features in figure 3 are associated with subcritical bifurcations.

These curves can be interpreted in terms of allowable noise in the system. Suppose that the system is designed to tolerate noise of amplitude ϵ_0 (e.g. $\epsilon_0 = \pm 0.1$). Then the finite-amplitude stability boundary is the curve corresponding to $\epsilon = \epsilon_0$. If the $\epsilon = \epsilon_0$ curve is on the stable side of the marginal-stability-boundary (subcritical) case, then the allowed noise in the system can trigger an instability if the finite-amplitude stability boundary is closer to the marginal-stability boundary than the $\epsilon = \epsilon_0$ curve. If the curve is on the unstable side (supercritical), operating between the marginal-stability boundary and the $\epsilon = \epsilon_0$ curve gives oscillations of amplitude less than ϵ_0, which are tolerable to the system. Taking these two arguments together, we see that the finite-amplitude stability boundary for an allowable noise level of ϵ_0 is the $\epsilon = \epsilon_0$ curve in parameter space.

One important qualitative implication of this work is in relation to the validity of linear stability analysis. If the $\epsilon =$ constant curves lie close to the marginal-stability boundary, then the marginal-stability boundary is generally valid in the sense that it will closely predict where a finite-amplitude instability is observed. To see this, first consider the subcritical case. If, for instance, the $\epsilon = 0.3$ curve lies close to the marginal-stability boundary, then a disturbance of amplitude $\pm 30\,\%$ will decay unless the parameter values are closer to the marginal-stability boundary than the $\epsilon = 0.3$ curve. Thus it would take a large disturbance to trigger an instability unless the parameter values were very close to the marginal-stability boundary.

For the supercritical case, the $\epsilon =$ constant curves correspond to the limit-cycle amplitude reached by the disturbance. If these curves lie close to the marginal-stability boundary, then, when parameter values are picked somewhat away from the marginal-stability boundary on the unstable side, a rather large-amplitude oscillation will result. In many cases, large-amplitude oscillations are as undesirable as oscillations that continue to grow, or excursions. Figure 4 indicates a situation in which the $\epsilon =$ constant curves lie close to the marginal-stability boundary. When this happens, the marginal-stability boundary gives a true representation of the operation stability boundary. On the other hand, figure 3 is an example in which the marginal-stability boundary may be invalid.

6. Conclusions

The main conclusion of the present analysis is that knowledge of the linear-stability boundary may be inadequate to assess the operating stability of process equipment involving boiling channels. Subcritical regions may exist on the stable side of the marginal-stability boundary. Thus caution should be exercised in designing boiling systems in order to avoid damaging instabilities excited by the presence of noise inherent in such systems.

It should be clear that the present analysis suggests that many new experimental studies are needed to test these analytical results, including unusual features (i.e.

'islands of instability') of the marginal-stability boundary, the frequency and amplitudes of limit-cycle oscillations in the supercritical cases, and the divergent instability characteristics of some subcritical regions in parameter space. Indeed, it is hoped that this study will stimulate needed experimental studies on finite-amplitude stability boundaries.

Appendix A

Here we show that if one encounters an inappropriate choice of critical parameter, while some other parameter is appropriate, then there are a large number of choices of appropriate critical parameters. If $\mu = \zeta_1 - \zeta_{1_0}$ is inappropriate to use at $\boldsymbol{\zeta}_0$, but $\nu = \zeta_j - \zeta_{j_0}$ is appropriate, then the variable μ^*, defined by

$$\mu = \mu^* \cos\theta, \quad \nu = \mu^* \sin\theta, \tag{A 1}$$

will be appropriate for any $\theta \neq n\pi$. Keeping θ constant, we have

$$\frac{\partial \alpha}{\partial \mu^*} = \frac{\partial \alpha}{\partial \mu}\cos\theta + \frac{\partial \alpha}{\partial \nu}\sin\theta, \tag{A 2}$$

which has a non-zero real part at $\boldsymbol{\zeta}_0$, since $\partial\alpha/\partial\nu$ has non-zero part, and $\sin\theta \neq 0$. Thus any linear combination of critical parameters is also a critical parameter, and furthermore, if one is appropriate, then among the combination there will be many appropriate choices of the critical parameter.

Let us consider a point $\boldsymbol{\zeta}^c(0)$ on the marginal-stability curve where μ is inappropriate to be selected as the critical parameter; μ is then arbitrary and ν undefined. In this case we must select a different critical parameter for the analysis near that point. Choosing simply another critical parameter corresponding to another 'good' control parameter ζ_j may be very lengthy because the whole computational procedure would have to be changed. So it is interesting to use the following approach, which operates on results obtained with the 'bad' control parameter $\boldsymbol{\zeta}^{(1)}$ obtained around $\boldsymbol{\zeta}^c(0)$. Specifically, we will try to build the hypersurface $\boldsymbol{\zeta}(\epsilon)$ off $\boldsymbol{\zeta}^c(0)$ by stretching from two points $\boldsymbol{\zeta}^{(1)}(0)$ and $\boldsymbol{\zeta}^{(2)}(0)$ close to $\boldsymbol{\zeta}^c(0)$ on the marginal-stability curve known by pieces of $\boldsymbol{\zeta}(\epsilon')$. From $\boldsymbol{\zeta}^{(1)}(0)$, $\boldsymbol{\zeta}(\epsilon')$ can be defined in terms of the 'bad' parameter as

$$\boldsymbol{\zeta}(\epsilon') = \boldsymbol{\zeta}^{(1)}(0) + (\mu_{(2)}\epsilon'^2 + \dots)\,\boldsymbol{e}_\mu. \tag{A 3}$$

Here $\boldsymbol{e}_\mu$ denotes the unit vector in the direction of increasing μ.

Let us define a new critical parameter $\hat{\mu}(\epsilon, \boldsymbol{\zeta})$ by trying to reach the same point $\boldsymbol{\zeta}$ in the hyperspace of parameters for the same value ϵ', but from the other neighbouring point $\zeta^{(2)}(0)$ on the marginal-stability curve:

$$\boldsymbol{\zeta}(\epsilon') = \boldsymbol{\zeta}^{(2)}(0) + (\hat{\mu}_{(2)}\epsilon'^2 + \dots)\,\boldsymbol{e}_{\hat{\mu}}. \tag{A 4}$$

Using (C 3) and (C 4) (Appendix C), we can determine $\hat{\mu}_{(2)}\,\boldsymbol{e}_{\hat{\mu}}$. Points on the hypersurface $\boldsymbol{\zeta}(\epsilon)$ can then be approximated by taking

$$\boldsymbol{\zeta}(\epsilon) = \boldsymbol{\zeta}^{(2)}_{(0)} + \epsilon^2 \hat{\mu}_{(2)}\,\boldsymbol{e}_{\hat{\mu}}. \tag{A 5}$$

Finally, points near $\boldsymbol{\zeta}^{(c)}_{(0)}$ can be found by taking judicious choices for $\boldsymbol{\zeta}^{(1)}_{(0)}$, $\boldsymbol{\zeta}^{(2)}_{(0)}$ and ϵ'.

The second mathematical result that we want to present is the Hopf theorem, which is discussed by Howard & Koeppel (1976) and Kazarinoff, Wan & Van Der Dreische (1978):

THEOREM: *There exists a $\epsilon_0 < 0$ such that, for each ϵ in the interval $-\epsilon_0 \leqslant \epsilon \leqslant \epsilon_0$, there exists a periodic solution $\boldsymbol{p}(t, \epsilon)$ of* (22), *with period $T(\epsilon)$. The parameter ϵ is related* *to μ by a functional relation $\mu = \mu(\epsilon)$, with $\mu(0) = 0, \boldsymbol{p}(t,0) = 0$, and $\boldsymbol{p}(t,\epsilon) \neq 0$ for all $\epsilon = 0$, and $T(0) = 2\pi/\omega_0$.*

These periodic solutions exist for exactly one of three cases. Either (i) only for $\mu > 0$ or (ii) only for $\mu < 0$ or (iii) only for $\mu = 0$. Furthermore, there is no other 'nearby' periodic solution, in the sense that for any $L > 0$ there are positive numbers a and b such that for $\mu < b$ there exists no periodic solution other than the steady-state solution $\delta \boldsymbol{X} \equiv 0$ and the periodic solution $\boldsymbol{p}(t,\epsilon)$ with period smaller than L, and whose amplitude is $|\delta \boldsymbol{X}(t)| < a$.

Appendix B

The equation for $j_{(1)}(\theta)$ is given by (43)–(45). The constants β_k and $b^k_{(0)}$ needed in (44) and (45) are given by

$$\beta_1 = \omega_0\{Fr^{-1}(1-e^{-N_{sub}\tilde{\tau}}) - N_{sub}\tilde{j} + \Lambda_0\tilde{j}^2(1-e^{N_{sub}\tilde{\tau}})\},$$

$$\beta_2 = \tilde{j}(1+\tilde{\tau})\omega_0, \quad \beta_3 = -N_{sub}\omega_0^2\tilde{j}\tilde{\tau},$$

$$\beta_4 = -Fr^{-1}(1-e^{-N_{sub}\tilde{\tau}}) + N_{sub}\tilde{j} - \Lambda_0\tilde{j}^2(1-e^{N_{sub}\tilde{\tau}}), \quad \beta_5 = -Fr^{-1},$$

$$\beta_6 = \omega_0 Fr^{-1}, \quad \beta_7 = Fr^{-1}e^{-N_{sub}\tilde{\tau}} + N_{sub}^2\tilde{j}\tilde{\tau} + \Lambda_0\tilde{j}^2 e^{N_{sub}\tilde{\tau}}(1+2N_{sub}\tilde{\tau}),$$

$$\beta_8 = -\tilde{j}\Lambda_0, \quad \beta_9 = \omega_0 Fr^{-1}e^{-N_{sub}\tilde{\tau}} + \Lambda_0\omega_0\tilde{j}^2 e^{N_{sub}\tilde{\tau}},$$

$$\beta_{10} = \frac{N_{sub}}{\omega_0}\{Fr^{-1}e^{-N_{sub}\tilde{\tau}} + \Lambda_0\tilde{j}^2 e^{N_{sub}\tilde{\tau}}\}, \quad \beta_{11} = -\Lambda_0\omega_0\tilde{j}^2, \quad \beta_{12} = -\frac{2\Lambda_0 N_{sub}\tilde{j}^2}{\omega_0},$$

$$\beta_{13} = -\frac{2\tilde{j}N_{sub}^2}{\omega_0}, \quad \beta_{14} = -N_{sub}^2\tilde{j}, \quad \beta_{15} = -\frac{4\Lambda_0 N_{sub}\tilde{j}^2}{\omega_0}, \quad \beta_{16} = -2\Lambda_0 N_{sub}\tilde{j}^2,$$

$$\beta_{17} = -\frac{\tilde{j}N_{sub}^3}{\omega_0^2}, \quad \beta_{18} = -2\Lambda_0\frac{N_{sub}^2\tilde{j}^2}{\omega_0^2}, \quad \beta_{19} = -\frac{2\Lambda_0 N_{sub}^2\tilde{j}^2}{\omega_0^2},$$

and

$$b^1_{(0)} = e^{-i\omega_0}, \quad b^2_{(0)} = i, \quad b^3_{(0)} = i\,e^{-i\omega_0}, \quad b^4_{(0)} = i\{e^{-i\omega_0}-1\},$$

$$b^5_{(0)} = i\,e^{-i\omega_0}\{e^{-i\omega_0\tilde{\tau}}-1\}, \quad b^6_{(0)} = e^{-i\omega_0}\{e^{-i\omega_0\tilde{\tau}}-1\}, \quad b^7_{(0)} = \frac{\omega_0 e^{-i\omega_0}}{N_{sub}-i\omega_0}\{e^{(N_{sub}-i\omega_0)\tilde{\tau}}-1\},$$

$$b^8_{(0)} = \frac{\omega_0 e^{-i\omega_0}}{2N_{sub}-i\omega_0}\{e^{(2N_{sub}-i\omega_0)\tilde{\tau}}-1\}, \quad b^9_{(0)} = \frac{i\omega_0 e^{-i\omega_0}}{N_{sub}-i\omega_0}\{e^{(N_{sub}-i\omega_0)\tilde{\tau}}-1\},$$

$$b^{10}_{(0)} = \frac{\omega_0^2 e^{-i\omega_0}}{N_{sub}-i\omega_0}\left\{e^{(N_{sub}-i\omega_0)\tilde{\tau}}\left[\tilde{\tau}-\frac{1}{N_{sub}-i\omega_0}\right]+\frac{1}{N_{sub}-i\omega_0}\right\},$$

$$b^{11}_{(0)} = \frac{i\omega_0 e^{-i\omega_0}}{2N_{sub}-i\omega_0}\{e^{(2N_{sub}-i\omega_0)\tilde{\tau}}-1\},$$

$$b^{12}_{(0)} = \frac{\omega_0^2 e^{-i\omega_0}}{2N_{sub}-i\omega_0}\left\{e^{(2N_{sub}-i\omega_0)\tilde{\tau}}\left[\tilde{\tau}-\frac{1}{2N_{sub}-i\omega_0}\right]+\frac{1}{2N_{sub}-i\omega_0}\right\},$$

$$b^{13}_{(0)} = \frac{\omega_0^2 e^{-i\omega_0}}{N_{sub}-i\omega_0}\left\{\frac{1}{N_{sub}-i\omega_0}[e^{(N_{sub}-i\omega_0)\tilde{\tau}}-1]-\tilde{\tau}\right\},$$

$$b^{14}_{(0)} = \frac{i\omega_0^2 e^{-i\omega_0}}{N_{sub}-i\omega_0}\left\{\frac{1}{N_{sub}-i\omega_0}[e^{(N_{sub}-i\omega_0)\tilde{\tau}}-1]-\tilde{\tau}\right\},$$

$$b_{(0)}^{15} = \frac{\omega_0^2\,\mathrm{e}^{-\mathrm{i}\omega_0}}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\left\{\frac{1}{2N_{\mathrm{sub}}-\mathrm{i}\omega_0}\,[\mathrm{e}^{(2N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1]+\frac{\tilde{\jmath}-1}{\tilde{\jmath}}\right\},$$

$$b_{(0)}^{16} = \frac{\mathrm{i}\omega_0^2\,\mathrm{e}^{-\mathrm{i}\omega_0}}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\left\{\frac{1}{2N_{\mathrm{sub}}-\mathrm{i}\omega_0}\,[\mathrm{e}^{(2N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1]+\frac{\tilde{\jmath}-1}{\tilde{\jmath}}\right\},$$

$$b_{(0)}^{17} = \frac{\omega_0^3\,\mathrm{e}^{-\mathrm{i}\omega_0}}{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)^3}\,\{\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}\,[\tilde{\tau}(N_{\mathrm{sub}}-\mathrm{i}\omega_0)-2]+2+\tilde{\tau}(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\},$$

$$b_{(0)}^{18} = \frac{\omega_0^3\,\mathrm{e}^{-\mathrm{i}\omega_0}}{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)}\left\{\frac{\mathrm{e}^{(2N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}}{2N_{\mathrm{sub}}-\mathrm{i}\omega_0}\left[\tilde{\tau}-\frac{1}{2N_{\mathrm{sub}}-\mathrm{i}\omega_0}\right]+\frac{1}{(2N_{\mathrm{sub}}-\mathrm{i}\omega_0)^2}\right.$$
$$\left.+\frac{1-\mathrm{e}^{(2N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}}{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\,(2N_{\mathrm{sub}}-\mathrm{i}\omega_0)}+\frac{1-\tilde{\jmath}}{\tilde{\jmath}(N_{\mathrm{sub}}-\mathrm{i}\omega_0)}\right\},$$

$$b_{(0)}^{19} = \frac{\omega_0^3\,\mathrm{e}^{-\mathrm{i}\omega_0}}{N_{\mathrm{sub}}(N_{\mathrm{sub}}-\mathrm{i}\omega_0)}\left\{\frac{N_{\mathrm{sub}}}{2N_{\mathrm{sub}}-\mathrm{i}\omega_0}\left[\mathrm{e}^{(2N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}\left(\tilde{\tau}-\frac{1}{2N_{\mathrm{sub}}-\mathrm{i}\omega_0}\right)\right.\right.$$
$$\left.\left.+\frac{1}{2N_{\mathrm{sub}}-\mathrm{i}\omega_0}\right]+\left[\frac{1-\tilde{\jmath}}{\tilde{\jmath}}-\tilde{\tau}\,\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}}\right]\right\}.$$

The nonlinear forcing function is given by

$$\begin{aligned}
\mathrm{Q}(\jmath_{(0)}) = {} & A_1\,\tau_{(0)}^2+A_2\,\tau_{(0)}\int_0^{\tilde{\tau}\omega_0}\mathrm{e}^{N_{\mathrm{sub}}\theta'/\omega_0}\jmath_{(0)}(\theta-\theta'-\omega_0)\,\mathrm{d}\theta' \\
& +A_3\int_0^{\tilde{\tau}\omega_0}\jmath_{(0)}(\theta-\theta'-\omega_0)\int_0^{\theta'}\mathrm{e}^{N_{\mathrm{sub}}\theta''/\omega_0}\jmath_{(0)}(\theta-\theta''-\omega_0)\,\mathrm{d}\theta''\,\mathrm{d}\theta' \\
& +A_4\int_0^{\tilde{\tau}\omega_0}\mathrm{e}^{N_{\mathrm{sub}}\theta'/\omega_0}\jmath_{(0)}(\theta-\theta'-\omega_0)\int_0^{\theta'}\mathrm{e}^{N_{\mathrm{sub}}\theta''/\omega_0}\jmath_{(0)}(\theta-\theta''-\omega_0)\,\mathrm{d}\theta''\,\mathrm{d}\theta' \\
& +A_5\int_0^{\tilde{\tau}\omega_0}\mathrm{e}^{N_{\mathrm{sub}}\theta'/\omega_0}\jmath_{(0)}(\theta-\theta'-\omega_0)\int_0^{\theta'}\jmath_{(0)}(\theta-\theta''-\omega_0)\,\mathrm{d}\theta''\,\mathrm{d}\theta' \\
& +A_6\int_0^{\tilde{\tau}\omega_0}\jmath_{(0)}(\theta-\theta'-\omega)\int_0^{\theta'}\jmath_{(0)}(\theta-\theta''-\omega_0)\,\mathrm{d}\theta''\,\mathrm{d}\theta' \\
& +A_7\int_0^{\tilde{\tau}\omega_0}\left(\int_0^{\theta'}\mathrm{e}^{N_{\mathrm{sub}}\theta'/\omega_0}\jmath_{(0)}(\theta-\theta''-\omega_0)\,\mathrm{d}\theta''\right)^2\mathrm{d}\theta' \\
& +A_8\,\jmath_{(0)}^2+A_9\,\jmath_{(0)}\,\lambda_{(0)}+A_{10}\,I_{0(0)}\jmath_{(0)}(\theta-\omega_0) \\
& +A_{11}\jmath_{(0)}\,I_{0(0)}+A_{12}\,\lambda_{(0)}\frac{\mathrm{d}\jmath_{(0)}}{\mathrm{d}\theta}+A_{13}\,I_{0(0)}\frac{\mathrm{d}\jmath_{(0)}}{\mathrm{d}\theta}+A_{14}\jmath_{(0)}\,J_{1(0)},
\end{aligned}$$

where the As are given by

$$A_1 = \tfrac{1}{2}\tilde{\jmath}N_{\mathrm{sub}}\,[Fr^{-1}-A_0\tilde{\jmath}^2\,\mathrm{e}^{2N_{\mathrm{sub}}\tilde{\tau}}],\quad A_2 = -\frac{\tilde{\jmath}N_{\mathrm{sub}}}{\omega_0}\,[N_{\mathrm{sub}}+2A_0\tilde{\jmath}\,\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}}],$$

$$A_3 = -\frac{N_{\mathrm{sub}}^2}{\omega_0^2},\quad A_4 = -\frac{2N_{\mathrm{sub}}\tilde{\jmath}A_0}{\omega_0^2},\quad A_5 = -A_4,$$

$$A_6 = A_4,\quad A_7 = \frac{N_{\mathrm{sub}}}{2\omega_0}A_4,\quad A_8 = -\tilde{\jmath}(1+\tilde{\tau})\,A_0,$$

$$A_9 = -2\tilde{\jmath}A_0,\quad A_{10} = -N_{\mathrm{sub}},\quad A_{11} = -2\tilde{\jmath}A_0,$$

$$A_{12} = -\omega_0,\quad A_{13} = A_{12},\quad A_{14} = -2A_0.$$

Appendix C

The constant and $\mathrm{e}^{\pm 2\mathrm{i}\theta}$ terms in the first-order solution are needed in the calculation of $\xi_{(2)}$ and $\mu_{(2)}$. If we write

$$y_{(1)} = a^{y}_{(1)} + b^{y}_{(1)}\,\mathrm{e}^{\mathrm{i}\theta} + \overline{b^{y}_{(1)}}\,\mathrm{e}^{-\mathrm{i}\theta} + c^{y}_{(1)}\,\mathrm{e}^{\mathrm{i}2\theta} + \overline{c^{y}_{(1)}}\,\mathrm{e}^{-\mathrm{i}2\theta}, \tag{C 1}$$

and further separate $a^{y}_{(1)}$ as

$$a^{y}_{(1)} = \tfrac{1}{2}(\hat{a}^{y}_{(1)} + \overline{\hat{a}^{y}_{(1)}}), \tag{C 2}$$

we have

$$\begin{aligned}
\hat{a}^{j}_{(1)} = \frac{1}{4\Phi(0)}\Big\{& A_1\,|b^{\tau}_{(0)}|^2 + A_2\, b^{\tau}_{(0)}\,\omega_0\,\mathrm{e}^{\mathrm{i}\omega_0}\frac{\mathrm{e}^{(N_{\mathrm{sub}}+\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}+\mathrm{i}\omega_0}\\
&+ A_3\frac{\omega_0}{N_{\mathrm{sub}}+\mathrm{i}\omega_0}\left[\omega_0\frac{1-\tilde{j}}{\tilde{j}} - \mathrm{i}\,(\mathrm{e}^{-\mathrm{i}\tilde{\tau}\omega_0}-1)\right]\\
&+ A_4\,\omega_0^2\left[\frac{\mathrm{e}^{2N_{\mathrm{sub}}\tilde{\tau}}-1}{2N_{\mathrm{sub}}(N_{\mathrm{sub}}+\mathrm{i}\omega_0)} - \frac{\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}^2+\omega_0^2}\right]\\
&- A_5\,\mathrm{i}\omega_0\left[\frac{1-\tilde{j}}{\tilde{j}} - \frac{\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\right] + A_6\,[-\mathrm{i}\tilde{\tau}\omega_0 - (\mathrm{e}^{-\mathrm{i}\tilde{\tau}\omega_0}-1)]\\
&+ A_7\frac{\omega_0^3}{N_{\mathrm{sub}}^2+\omega_0^2}\left\{\frac{\mathrm{e}^{2N_{\mathrm{sub}}\tilde{\tau}}-1}{2N_{\mathrm{sub}}} - \frac{\mathrm{e}^{(N_{\mathrm{sub}}+\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}+\mathrm{i}\omega_0} - \frac{\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0} + \tilde{\tau}\right\}\\
&+ A_8 + A_9\,\overline{b^{\lambda}_{(0)}} - A_{10}\,b^{I_0}_{(0)}\,\mathrm{e}^{\mathrm{i}\omega_0} + A_{11}\,\overline{b^{I_0}_{(0)}} - A_{12}\,b^{\lambda}_{(0)}\,\mathrm{i} - A_{13}\,\overline{b^{I_0}_{(0)}}\,\mathrm{i} + A_{14}\,\overline{b^{J_1}_{(0)}}\Big\}
\end{aligned} \tag{C 3}$$

and

$$\begin{aligned}
c^{j}_{(1)} = \frac{1}{4\Phi(\mathrm{i}2\omega_0)}\Big\{& A_1(b^{\tau}_{(0)})^2 + A_2\,b^{\tau}_{(0)}\,\omega_0\,\mathrm{e}^{-\mathrm{i}\omega_0}\frac{\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\\
&+ A_3\frac{\omega_0\,\mathrm{e}^{-2\mathrm{i}\omega_0}}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\left[\omega_0\frac{\mathrm{e}^{(N_{\mathrm{sub}}-2\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-2\mathrm{i}\omega_0} - \mathrm{i}\,(\mathrm{e}^{-\mathrm{i}\tilde{\tau}\omega_0}-1)\right]\\
&+ A_4\frac{\omega_0^2\,\mathrm{e}^{-2\mathrm{i}\omega_0}}{2(N_{\mathrm{sub}}-\mathrm{i}\omega_0)^2}\,[\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1]^2\\
&+ A_5\,\mathrm{i}\omega_0\,\mathrm{e}^{-2\mathrm{i}\omega_0}\left[\frac{\mathrm{e}^{(N_{\mathrm{sub}}-2\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-2\mathrm{i}\omega_0} - \frac{\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\right] - \tfrac{1}{2}A_6\,\mathrm{e}^{-2\mathrm{i}\omega_0}\,[\mathrm{e}^{-\mathrm{i}\tilde{\tau}\omega_0}-1]^2\\
&+ A_7\frac{\omega_0^2\,\mathrm{e}^{-2\mathrm{i}\omega_0}}{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)^2}\left[\frac{\mathrm{e}^{2(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1}{2(N_{\mathrm{sub}}-\mathrm{i}\omega_0)} - \frac{2[\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}\omega_0)\tilde{\tau}}-1]}{N_{\mathrm{sub}}-\mathrm{i}\omega_0} - \tilde{\tau}\right]\\
&+ A_8 + A_9\,b^{\lambda}_{(0)} + A_{10}\,b^{I_0}_{(0)}\,\mathrm{e}^{-\mathrm{i}\omega_0} + A_{11}\,b^{I_0}_{(0)} + A_{12}\,b^{\lambda}_{(0)}\,\mathrm{i} + A_{13}\,b^{I_0}_{(0)}\,\mathrm{i} + A_{14}\,b^{J_1}_{(0)}\Big\},
\end{aligned} \tag{C 4}$$

where Φ is given by

$$\begin{aligned}
\Phi(0) = \alpha_1 + \alpha_2 + \alpha_3\,\omega_0 + \alpha_4\,\tilde{\tau}\omega_0 + \alpha_5\,\omega_0\frac{1-\tilde{j}}{\tilde{j}} + \alpha_6\,\omega_0\frac{\mathrm{e}^{2N_{\mathrm{sub}}}-1}{2N_{\mathrm{sub}}}\\
+ \alpha_7\frac{\omega_0^2}{N_{\mathrm{sub}}}\left[\frac{1-\tilde{j}}{\tilde{j}} - \tilde{\tau}\right] + \alpha_8\frac{\omega_0^2}{2}\left[\frac{1-\tilde{j}}{\tilde{j}}\right]^2
\end{aligned} \tag{C 5}$$

and

$$\Phi(\mathrm{i}2\omega_0) = \alpha_0 2\mathrm{i}\omega_0+\alpha_1+\alpha_2\,\mathrm{e}^{-\mathrm{i}2\omega_0}+\tfrac{1}{2}\alpha_3\,\mathrm{i}(\mathrm{e}^{-\mathrm{i}2\omega_0}-1)+\tfrac{1}{2}\alpha_4\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}2\omega_0}(\mathrm{e}^{-2\mathrm{i}\omega_0\tilde{\tau}}-1)$$
$$+\alpha_5\,\omega_0\,\mathrm{e}^{-\mathrm{i}2\omega_0}\frac{\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}2\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}2\omega_0}+\tfrac{1}{2}\alpha_6\,\omega_0\,\mathrm{e}^{-\mathrm{i}2\omega_0}\frac{\mathrm{e}^{2(N_{\mathrm{sub}}-\mathrm{i}2\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}$$
$$+\alpha_7\frac{\omega_0^2\,\mathrm{e}^{-\mathrm{i}2\omega_0}}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\,\frac{\mathrm{e}^{(N_{\mathrm{sub}}-\mathrm{i}2\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}2\omega_0}+\alpha_8\frac{\omega_0^2\,\mathrm{e}^{-\mathrm{i}2\omega_0}}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\left[\frac{\mathrm{e}^{2(N_{\mathrm{sub}}-\mathrm{i}2\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}-\frac{1-\tilde{\jmath}}{\tilde{\jmath}}\right], \quad \text{(C 6)}$$

where the αs are given by

$$\left.\begin{aligned}&\alpha_0=\tilde{\jmath}(1+\tilde{\tau}),\quad \alpha_1=2\Lambda\tilde{\jmath},\quad \alpha_2=N_{\mathrm{sub}}\,\tilde{\tau}\tilde{\jmath}\\&\alpha_3=Fr^{-1}(1-\mathrm{e}^{-N_{\mathrm{sub}}\tilde{\tau}})-N_{\mathrm{sub}}\tilde{\jmath}+\Lambda\tilde{\jmath}(1-\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}}),\quad \alpha_4=Fr^{-1},\\&\alpha_5=-Fr^{-1}\,\mathrm{e}^{-N_{\mathrm{sub}}\tilde{\tau}}-\Lambda\tilde{\jmath}^2\,\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}},\quad \alpha_6=\Lambda\tilde{\jmath}^2,\quad \alpha_7=N_{\mathrm{sub}}^2\tilde{\jmath},\quad \alpha_8=2\Lambda N_{\mathrm{sub}}\tilde{\jmath}^2.\end{aligned}\right\}\quad \text{(C 7)}$$

The remaining coefficients can be calculated from $a^{j}_{(1)}$ and $c^{j}_{(1)}$, and are given by

$$a^{\lambda}_{(1)} = a^{j}_{(1)}, \quad \text{(C 8)}$$

$$c^{\lambda}_{(1)} = \frac{\mathrm{i}}{2\omega_0}(\mathrm{e}^{-2\mathrm{i}\omega_0}-1)\,c^{j}_{(1)}, \quad \text{(C 9)}$$

$$a^{\tau}_{(1)} = \frac{\mathrm{e}^{-N_{\mathrm{sub}}\tilde{\tau}}}{\tilde{\jmath}}\left\{a^{\lambda}_{(1)}+a^{j}_{(1)}\frac{(1-\tilde{\jmath})}{\tilde{\jmath}}\right\}-\tfrac{1}{8}N_{\mathrm{sub}}\,|b^{\tau}_{(0)}|^2-\frac{b^{\tau}_{(0)}\,\mathrm{e}^{\mathrm{i}\omega_0(1+\tilde{\tau})}}{4\tilde{\jmath}} \quad \text{(C 10)}$$

$$c^{\tau}_{(1)} = -\frac{\mathrm{e}^{-N_{\mathrm{sub}}\tilde{\tau}}}{\tilde{\jmath}}\left\{c^{\lambda}_{(1)}+c^{j}_{(1)}\,\mathrm{e}^{-2\mathrm{i}\omega_0}\frac{\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}-2\mathrm{i}\omega_0\tilde{\tau}}-1}{N_{\mathrm{sub}}-2\mathrm{i}\omega_0}\right\}-\left\{\tfrac{1}{8}N_{\mathrm{sub}}\,(b^{\tau}_{(0)})^2+\frac{b^{\tau}_{(0)}\,\mathrm{e}^{-\mathrm{i}\omega_0(1+\tilde{\tau})}}{4\tilde{\jmath}}\right\} \quad \text{(C 11)}$$

$$a^{I_0}_{(1)} = \tilde{\jmath}a^{\tau}_{(1)}+a^{j}_{(1)}\,\tilde{\tau}+\tfrac{1}{4}\,\mathrm{e}^{-\mathrm{i}\omega_0(1+\tilde{\tau})}\,\overline{b^{\tau}_{(0)}} \quad \text{(C 12)}$$

$$c^{I_0}_{(1)} = \tilde{\jmath}c^{\tau}_{(1)}+c^{j}_{(1)}\,\mathrm{i}\,\mathrm{e}^{-2\mathrm{i}\omega_0}\frac{\mathrm{e}^{-2\mathrm{i}\tilde{\tau}\omega_0}-1}{2\omega_0}+\tfrac{1}{4}\,\mathrm{e}^{-\mathrm{i}\omega_0(1+\tilde{\tau})}\,b^{\tau}_{(0)}, \quad \text{(C 13)}$$

$$a^{J}_{(1)} = \tilde{\jmath}(1-\tilde{\jmath})\,a^{\tau}_{(1)}+\frac{2\tilde{\jmath}a^{j}_{(1)}}{N_{\mathrm{sub}}}\left\{\frac{1-\tilde{\jmath}}{\tilde{\jmath}}-\tilde{\tau}\right\}+\tfrac{1}{8}\tilde{\jmath}^2\,\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}}\,|b^{\tau}_{(0)}|^2$$
$$+\tfrac{1}{4}(1-\tilde{\jmath})\,b^{\tau}_{(0)}\,\mathrm{e}^{\mathrm{i}\omega_0(1+\tilde{\tau})}+\tfrac{1}{4}\tilde{\jmath}\,\mathrm{e}^{-\mathrm{i}\omega_0}\,\mathrm{i}\,\frac{\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}-\mathrm{i}\omega_0\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\,\overline{b^{\tau}_{(0)}}$$
$$+\frac{1}{4}\,\frac{1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\left\{\frac{1-\tilde{\jmath}}{\tilde{\jmath}}+\mathrm{i}\,(\mathrm{e}^{\mathrm{i}\tilde{\tau}\omega_0}-1)\right\}, \quad \text{(C 14)}$$

$$c^{J}_{(1)} = \tilde{\jmath}(1-\tilde{\jmath})\,c^{\tau}_{(1)}+\frac{\tilde{\jmath}}{N_{\mathrm{sub}}}c^{j}_{(1)}\,\mathrm{e}^{-2\mathrm{i}\omega_0}\left\{\frac{\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}-2\mathrm{i}\omega_0\tilde{\tau}}-1}{N_{\mathrm{sub}}-2\mathrm{i}\omega_0}+\tfrac{1}{2}\mathrm{i}(1-\mathrm{e}^{-2\mathrm{i}\omega_0\tilde{\tau}})\right\}$$
$$+\tfrac{1}{8}\tilde{\jmath}^2\,\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}}(b^{\tau}_{(0)})^2+\tfrac{1}{4}(1-\tilde{\jmath})\,b^{\tau}_{(0)}\,\mathrm{e}^{-\mathrm{i}\omega_0(1+\tilde{\tau})}$$
$$+\tfrac{1}{4}\,\tilde{\jmath}^2\,b^{\tau}_{(0)}\,\mathrm{e}^{-\mathrm{i}\omega_0}\frac{\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}-\mathrm{i}\omega_0\tilde{\tau}}-1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}$$
$$+\frac{1}{4}\,\frac{1}{N_{\mathrm{sub}}-\mathrm{i}\omega_0}\left\{\frac{\mathrm{e}^{(N_{\mathrm{sub}}-2\mathrm{i}\omega_0)\tilde{\tau}}-1}{N_{\mathrm{sub}}-2\mathrm{i}\omega_0}-\frac{\mathrm{i}\,(\mathrm{e}^{-\mathrm{i}\tilde{\tau}\omega_0}-1)}{2\omega_0}\right\}. \quad \text{(C 15)}$$

Forcing function

The forcing function, which gives $\mathrm{e}^{\mathrm{i}\theta}$ contributions in the second-order equation for $j_{(2)}$, is given by (49), where we write $S = S_Q + S_C$. These coefficients are given by

$$
\begin{aligned}
S_Q = {} & 2\tilde{\jmath}(1+\tilde{\tau})\Lambda_0\{a^{\jmath}_{(1)} + c^{\jmath}_{(1)}\} - 2\tilde{\jmath}\Lambda_0\{a^{\lambda}_{(1)} + c^{\lambda}_{(1)} + c^{\jmath}_{(1)}\overline{b^{\lambda}_{(0)}} + a_{(1)}b^{\lambda}_{(0)}\} \\
& - N_{\mathrm{sub}}\{c^{I_0}_{(1)}\,\mathrm{e}^{\mathrm{i}\omega_0} + a^{I_0}_{(1)}\,\mathrm{e}^{-\mathrm{i}\omega_0} + a^{\jmath}_{(1)}b^{I_0}_{(0)} + c^{\jmath}_{(1)}\,\mathrm{e}^{-\mathrm{i}2\omega_0}\overline{b^{I_0}_{(0)}}\} \\
& - 2\tilde{\jmath}\Lambda_0\{a^{I_0}_{(1)} + c^{I_0}_{(1)} + c^{\jmath}_{(1)}\overline{b^{I_0}_{(0)}} + a^{\jmath}_{(1)}b^{I_0}_{(0)}\} \\
& - 2\Lambda_0\{a^{J_1}_{(1)} + c^{J_1}_{(1)} + c^{\jmath}_{(1)}\overline{b^{J_1}_{(0)}} + a^{\jmath}_{(1)}b^{J_1}_{(0)}\} - 2\mathrm{i}\omega_0 c^{\jmath}_{(1)}(\overline{b^{\lambda}_{(0)}} + \overline{b^{I_0}_{(0)}}) \\
& - \mathrm{i}\omega_0\{a^{\lambda}_{(1)} + a^{I_0}_{(1)} - c^{\lambda}_{(1)} - c^{I_0}_{(1)}\} \\
& + \tilde{\jmath}N_{\mathrm{sub}}(Fr^{-1} - \tilde{\jmath}^2\Lambda_0\,\mathrm{e}^{2N_{\mathrm{sub}}\tilde{\tau}})\{c^{\tau}_{(1)}\overline{b^{\tau}_{(0)}} + a^{\tau}_{(1)}b^{\tau}_{(0)}\} + \frac{1}{\omega_0}\{N^2_{\mathrm{sub}} + 2N_{\mathrm{sub}}\tilde{\jmath}\Lambda_0\,\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}}\} \\
& \times\left\{-\omega_0(1-\tilde{\jmath})\,a^{\jmath}_{(1)}\overline{b^{\tau}_{(0)}} - \frac{\tilde{\jmath}\omega_0\,\mathrm{e}^{-\mathrm{i}2\omega_0}}{N_{\mathrm{sub}} - \mathrm{i}2\omega_0}[\mathrm{e}^{(N_{\mathrm{sub}} - \mathrm{i}2\omega_0)\tilde{\tau}} - 1]\,c^{\jmath}_{(1)} - \tilde{\jmath}a^{\tau}_{(1)}b^{\jmath}_{(0)} - \tilde{\jmath}c^{\tau}_{(1)}\overline{b^{\jmath}_{(0)}}\right\} \\
& + \frac{1}{\omega_0}\{N^2_{\mathrm{sub}} + 2N_{\mathrm{sub}}\tilde{\jmath}\Lambda_0\}\left\{-\frac{\omega_0\,\mathrm{e}^{-\mathrm{i}\omega_0}}{N_{\mathrm{sub}}}a^{\jmath}_{(1)}\left[\frac{\mathrm{e}^{(N_{\mathrm{sub}} - \mathrm{i}\omega_0)\tilde{\tau}} - 1}{N_{\mathrm{sub}} - \mathrm{i}\omega_0} + \frac{\mathrm{i}}{\omega_0}(1 - \mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}})\right]\right. \\
& \left. - \frac{\omega_0\,\mathrm{e}^{-\mathrm{i}\omega_0}}{N_{\mathrm{sub}} - \mathrm{i}2\omega_0}c^{\jmath}_{(1)}\left[\frac{\mathrm{e}^{(N_{\mathrm{sub}} - \mathrm{i}\omega_0)\tilde{\tau}} - 1}{N_{\mathrm{sub}} - \mathrm{i}\omega_0} + \frac{\mathrm{i}}{\omega_0}(\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}} - 1) - \frac{\mathrm{e}^{(N_{\mathrm{sub}} - 2\mathrm{i}\omega_0)\tilde{\tau}} - 1}{N_{\mathrm{sub}} - \mathrm{i}2\omega_0} + \tilde{\tau}\right]\right\}
\end{aligned}
\tag{C 16}
$$

and

$$
\begin{aligned}
4S_C = {} & -\Lambda_0\{2b^{\lambda}_{(0)} + \overline{b^{\lambda}_{(0)}} + 2b^{I_{(0)}}_{(0)} + \overline{b^{I_{(0)}}_{(0)}}\} + \tfrac{1}{2}\tilde{\jmath}N^2_{\mathrm{sub}}\{Fr^{-1} - 3\Lambda_0\tilde{\jmath}^2\,\mathrm{e}^{2N_{\mathrm{sub}}\tilde{\tau}}\}\overline{b^{\tilde{\tau}}_{(0)}}\,(b^{\tilde{\tau}}_{(0)})^2 \\
& + \tfrac{1}{2}N_{\mathrm{sub}}\{Fr^{-1} - N_{\mathrm{sub}}\tilde{\jmath}\,\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}} - 3\Lambda_0\tilde{\jmath}^2\,\mathrm{e}^{2N_{\mathrm{sub}}\tilde{\tau}}\}\{2b^{\tau}_{(0)}\overline{b^{\tau}_{(0)}}\,\mathrm{e}^{-\mathrm{i}\omega_0} + (b^{\tau}_{(0)})^2\,\mathrm{e}^{\mathrm{i}\omega_0}\} \\
& - \frac{1}{\omega_0}\{N^2_{\mathrm{sub}} + 2\Lambda_0\tilde{\jmath}\,\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}}\}\{\overline{b^{\jmath}_{(0)}}\,b^{\tau}_{(0)}\,\mathrm{e}^{-\mathrm{i}\omega_0(1+\tilde{\tau})} + b^{\jmath}_{(0)}b^{\tau}_{(0)}\,\mathrm{e}^{-\mathrm{i}\omega_0(1+\tilde{\tau})} + b^{\jmath}_{(0)}b^{\tau}_{(0)}\,\mathrm{e}^{\mathrm{i}\omega_0(1+\tilde{\tau})}\} \\
& - \frac{N^2_{\mathrm{sub}}\Lambda_0\tilde{\jmath}^2}{\omega_0}\,\mathrm{e}^{N_{\mathrm{sub}}\tilde{\tau}}\{2b^{\tau}_{(0)}\overline{b^{\tau}_{(0)}}\,b^{\jmath}_{(0)} + (b^{\tau}_{(0)})^2\,\overline{b^{\jmath}_{(0)}}\} \\
& - \frac{N^2_{\mathrm{sub}}\Lambda_0\tilde{\jmath}^2}{\omega_0^2}\{2b^{\jmath}_{(0)}\overline{b^{\jmath}_{(0)}}\,b^{\tau}_{(0)} + (b^{\jmath}_{(0)})^2\,\overline{b^{\tau}_{(0)}}\Big\} \\
& - \frac{N^2_{\mathrm{sub}}\Lambda_0\,\mathrm{e}^{-\mathrm{i}\omega_0}}{N_{\mathrm{sub}} - \mathrm{i}\omega_0}\left\{\frac{(\mathrm{e}^{(2N_{\mathrm{sub}} - \mathrm{i}\omega_0)\tilde{\tau}} - 1)(3N_{\mathrm{sub}} - \mathrm{i}\omega_0)}{(2N_{\mathrm{sub}} - \mathrm{i}\omega_0)(N^2_{\mathrm{sub}} + \omega_0^2)}\right. \\
& \left. - \frac{4N_{\mathrm{sub}}(1-\tilde{\jmath})}{\tilde{\jmath}(N^2_{\mathrm{sub}} + \omega_0^2)} - \frac{\mathrm{e}^{(N_{\mathrm{sub}} - 2\mathrm{i}\omega_0)\tilde{\tau}} - 1}{(N_{\mathrm{sub}} - 2\mathrm{i}\omega_0)(N_{\mathrm{sub}} + \mathrm{i}\omega_0)} + \frac{2\mathrm{i}\,(\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}} - 1)}{\omega_0(N_{\mathrm{sub}} + \mathrm{i}\omega_0)} + \frac{\mathrm{i}(\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}} - 1)}{(N_{\mathrm{sub}} - \mathrm{i}\omega_0)\,\omega_0}\right\}.
\end{aligned}
\tag{C 17}
$$

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